

HIGHLIGHT OF THE MONTH

ON OSCILLATIONS OF THERMODYNAMIC QUANTITIES FOR DEGENERATE FERMI GAS AT LOW TEMPERATURES

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Abstract

The electron energy level quantization in a magnetic field leads to oscillations of thermodynamical quantities for the degenerate electron Fermi gas and is the cause of a periodical dependence of the magnetic susceptibility of the electron gas on the magnetic field (de Haas – van Alphen effect). It is obvious that an analogous oscillatory dependence for the degenerate Fermi gas of particles with discrete quantum energy levels on parameters which determine the energy level positions appears always when the quantization of energy levels takes place (for example, bounded volume of the gas or external field with a spatially increasing potential).

It is known that the quantization of energy levels of electrons in a magnetic field leads to oscillations of thermodynamic quantities for a degenerate electron Fermi gas and causes a periodic dependence of magnetic susceptibility of the electron gas on the field strength (de Haas-van Alphen effect) [1]. Obviously, an analogous effect of the oscillating dependence of thermodynamic quantities for a degenerate Fermi gas of particles with quantized energy levels on those parameters which determine the position of these levels will take place always when the causes leading to the quantization of energy levels exist (e. g. finite volume, external fields with an increasing potential, etc.).

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Below we calculate the oscillating terms in the thermodynamic potentials causing the whole effect under quite general assumptions regarding the energy spectrum of the particles (or quasiparticles) forming the ideal degenerate Fermi gas.

1 Calculation of oscillating part of thermodynamic potential Ω

Let us suppose that the "classical" problem of the particle motion is solved. Then the energy \mathcal{E} of the particle can be expressed by means of action variables I_i :

$$\mathcal{E} = \mathcal{E}(I_1, I_2, I_3) \quad (1)$$

As the oscillations of thermodynamic quantities at low temperatures are determined by the energy spectrum near the Fermi energy, we shall be interested only in the high energy levels and therefore we can apply the quasiclassical approximation to the quantization of the particle motion. The quasiclassical quantization of a system with several degrees of freedom performing finite motion is given by [2]:

$$I_i = (n_i + \gamma_i)h_i ; \quad 0 < \gamma_i < 1 , \quad (2)$$

and the quantized energy levels E are obtained from (1) by substitution of $(n_i + \gamma_i)h$ for I_i :

$$E(\mathbf{n}) = E[(n_1 + \gamma_1)h, (n_2 + \gamma_2)h, (n_3 + \gamma_3)h]$$

(here \mathbf{n} means a vector with integer components n_1, n_2, n_3).

The thermodynamic potential Ω relevant for the calculation of thermodynamic quantities is given by

$$\Omega = -\theta \sum_{\mathbf{n}} \ln \left\{ 1 + e^{[\zeta - E(\mathbf{n})]/\theta} \right\} ,$$

where $\theta = kT$, ζ denotes the chemical potential and the sum extends over all possible values of \mathbf{n} .

To get the oscillating part of Ω we use a three-dimensional analogy of Poisson formula where the sum over \mathbf{n} is replaced by the integral:

$$\begin{aligned} \Omega &= \int f[\zeta - E(\mathbf{n})] d\mathbf{n} + 2\text{Re} \sum_{\mathbf{k}} \mathcal{L}_{\mathbf{k}} = \\ &= \frac{1}{h^2} \int f[\zeta - \mathcal{E}(I_1, I_2, I_3)] dI_1 dI_2 dI_3 + 2\text{Re} \sum_{\mathbf{k}} \mathcal{L}_{\mathbf{k}} , \end{aligned} \quad (3)$$

where

$$f(x) = -\theta \ln(1 + e^{x/\theta})$$

The vector \mathbf{k} denotes a triplet of integer indices k_1, k_2, k_3 and the sum over \mathbf{k} extends over all non-negative k_1, k_2, k_3 excluding $\mathbf{k} = 0$. The symbol $\mathcal{L}_{\mathbf{k}}$ stands for the following integral:

$$\mathcal{L}_{\mathbf{k}} = \int f([\zeta - E(\mathbf{n})])e^{2\pi i \mathbf{k} \mathbf{n}} d\mathbf{n}.$$

The oscillating part of Ω is contained in the second term of the expression (3).

Before we calculate the integrals $\mathcal{L}_{\mathbf{k}}$ entering Eq. (3) we note that the integration over \mathbf{n} is carried out only over the first octant (from $n_i = 0$ to $n_i = \infty$). As a consequence and due to the discontinuity of the integrand at the boundary of the region of integration, the result will contain two-dimensional and one-dimensional integrals besides the three-dimensional integrals $\mathcal{L}_{\mathbf{k}}$. Hence, the formula for the oscillating part of Ω reads:

$$\begin{aligned} \Omega_{\text{osc}} &= 2\text{Re} \sum_{\mathbf{k}} \mathcal{L}_{\mathbf{k}} + \text{Re} \left\{ \sum'_{k_1, k_2} P_{k_1, k_2} + \sum'_{k_1, k_3} P_{k_1, k_3} + \sum'_{k_2, k_3} P_{k_2, k_3} \right\} + \\ &+ \frac{1}{2} \text{Re} \left\{ \sum'_{k_1} Q_{k_1} + \sum'_{k_2} Q_{k_2} + \sum'_{k_3} Q_{k_3} \right\}, \end{aligned} \quad (4)$$

where the two-dimensional integral P_{k_1, k_2} is given by :

$$P_{k_1, k_2} = \int \int f[\zeta - E(n_1, n_2, 0)] e^{2\pi i (k_1 n_1 + k_2 n_2)} dn_1 dn_2,$$

P_{k_1, k_3} and P_{k_2, k_3} are given by analogy. The one-dimensional integrals $Q_{k_1}, Q_{k_2}, Q_{k_3}$ are given by expressions of the form

$$Q_{k_1} = \int f[\zeta - E(n_1, 0, 0)] e^{2\pi i k_1 n_1} dn_1$$

Primes at the sum symbols in Eq. (4) means that the term with $\mathbf{k} = 0$ ($k_1 = k_2 = k_3 = 0$) is excluded.

Obviously, the integral $\mathcal{L}_{\mathbf{k}}$ can be transformed to

$$\mathcal{L}_{\mathbf{k}} = \int f(\zeta - E) \left\{ \int_{\substack{E(\mathbf{n}) = E \\ n_i \geq 0}} \frac{e^{2\pi i \mathbf{k} \mathbf{n}} d\Omega_{\mathbf{n}}}{|\nabla E(\mathbf{n})|} \right\} dE,$$

where $d\Omega_{\mathbf{n}}$ denotes an element of the constant energy surface

$$E(\mathbf{n}) = E, \quad (5)$$

in the direction \mathbf{n} . The integration in the surface integral inside the braces is restricted to that part of the constant energy surface (5) which lies in the first octant.

Employing the starting assumption regarding the large quantum numbers ($\mathbf{k} \mathbf{n} \gg 1$), an asymptotic estimate of the integral $\mathcal{L}_{\mathbf{k}}$ can be obtained [3]:

$$\mathcal{L}_{\mathbf{k}} \approx \int dE f(\zeta - E) \sum_{\nu} \frac{e^{2\pi i \mathbf{k} \mathbf{n}_{\nu} \pm i\pi/4 \pm i\pi/4}}{|\mathbf{k}| |\nabla E(\mathbf{n}_{\nu})| \sqrt{|K_{\nu}(E)|}};$$

where $\mathbf{n}_\nu = \mathbf{n}_\nu(\mathbf{k}, E)$ denotes the radius vector in the space \mathbf{n} of those points on the surface (5) at which the direction of the surface normal is parallel to \mathbf{k} ; $K_\nu(E)$ is the Gaussian curvature of the surface at these points. The sum extends over all such points in the first octant. The phases $\nu\pi/4$ enter the argument of the exponential with the same sign if the point \mathbf{n}_ν is elliptical. They enter with minus sign if the convexity of the surface at a given point is directed along \mathbf{k} and they enter with plus sign if the convexity is directed in the opposite direction. In case of a hyperbolic point the phases $\nu\pi/4$ enter the exponent with different signs.

Assuming $\partial n_\nu/\partial E \neq 0$ and an obvious condition $\theta \ll \zeta$ the main contribution to the oscillating part of $\mathcal{L}_\mathbf{k}$ comes from integration over the neighbourhood of the sharp non-stationarity of the Fermi distribution function df/dE , i. e. near $E = \zeta$. Taking this into account, we expand $\mathbf{n}_\nu(\mathbf{k}, E)$ in the argument of exponential into powers of $E - \zeta$ and integrate:

$$(\mathcal{L}_\mathbf{k})_{\text{osc}} \approx \frac{1}{(2\pi)^2 \sqrt{k_1^2 + k_2^2 + k_3^2}} \sum_\nu \frac{\Psi(\lambda_\mathbf{k}) e^{2\pi i \mathbf{k} \mathbf{n}_\nu(k, \zeta) \pm i\pi/4 \pm i\pi/4}}{(\mathbf{k}, \partial n_\nu/\partial \zeta)^2 |\nabla E(\mathbf{n}_\nu)| \sqrt{|K_\nu(\zeta)|}},$$

$$\Psi(\lambda) = \lambda / \sinh \lambda ; \quad \lambda_\mathbf{k} = 2\pi^2 (\mathbf{k}, \partial \mathbf{n}_\nu/\partial \zeta) \theta$$

The two-dimensional and one-dimensional integrals can be evaluated similarly. The final results are:

$$(P_{k_1, k_2})_{\text{osc}} \approx \frac{1}{(2\pi)^2 (k_1^2 + k_2^2)^{\frac{1}{4}}} \times$$

$$\times \sum_\mu \frac{\Psi(\lambda_{k_1, k_2}) e^{2\pi i (k_1 n_{\mu 1} + k_2 n_{\mu 2}) \pm i\pi/4}}{\left(k_1 \frac{\partial n_{\mu 1}}{\partial \zeta} + k_2 \frac{\partial n_{\mu 2}}{\partial \zeta}\right)^2 \left[\left(\frac{\partial E}{\partial n_1}\right)^2 + \left(\frac{\partial E}{\partial n_2}\right)^2\right]^{\frac{1}{2}} \sqrt{|K_{12\mu}(\zeta)|}};$$

$$\lambda_{k_1, k_2} = 2\pi^2 \theta \left(k_1 \frac{\partial n_{\mu 1}}{\partial \zeta} + k_2 \frac{\partial n_{\mu 2}}{\partial \zeta}\right);$$

here $n_{\mu i}(\zeta) = n_{\mu i}(k_1, k_2, \zeta)$ denotes the coordinates of those points on the plain curve

$$E(n_1, n_2, 0) = \zeta \tag{6}$$

at which the direction of the curve normal is parallel to the two-dimensional vector (k_1, k_2) ; $K_{12\mu}(\zeta)$ stands for the curvature of the curve (6) at these points and the sum extends over all such points in the first quadrant. The phase $\nu\pi/4$ enters the exponential with minus sign if the curvature of the curve (6) at the point $(n_{\mu 1}, n_{\mu 2})$ is directed along (k_1, k_2) and it enters with plus sign if the curvature of the curve is in the opposite direction. Further:

$$(Q_{k_1})_{\text{osc}} \approx \frac{\Psi(\lambda_{k_1})}{(2\pi k)^2 \partial n_1/\partial \zeta} e^{2\pi i k_1 n_1(\zeta)};$$

$$\lambda_{k_1} = 2\pi^2 \theta \left(k_1 \frac{\partial n_1}{\partial \zeta}\right);$$

where $n_1(\zeta)$ denotes this value of n_1 for which $E(n_1, 0, 0) = \zeta$. The formulas for P_{k_1, k_3} , P_{k_3, k_2} and Q_{k_2} , Q_{k_3} have an analogous form. Employing the obtained asymptotic expressions for $\mathcal{L}_{\mathbf{k}}$, P and Q in (4) we get the oscillating part of Ω . It comes out that Ω_{osc} contains several terms, each of them having its own period of oscillations. The periods of oscillations in individual terms of Ω_{osc} can be determined from the knowledge of the dependence of the arguments of the corresponding exponentials on the parameters determining the position of energy levels. It should be noted that if the ratio of periods of different terms can be in general arbitrary, the oscillations can have a very complicated form in case of several terms of the same order of magnitude in (4).

The solution of the "classical" problem to find the energy of a particle in the form (1) is connected in general with well-known difficulties and can be performed easily only for separating variables.

We shall give the simplest examples in which it is possible to calculate the oscillating part of the thermodynamic potentials to the very end and to obtain the periods of corresponding oscillations.

2 Electron gas in three-dimensional potential well

Let us consider the electron gas with an arbitrary law of dispersion:

$$\mathcal{E} = \mathcal{E}(p_1, p_2, p_3) \quad (7)$$

in an infinitely high rectangular potential well with sizes L_1 , L_2 and L_3 along the respective axes. As it is well-known, the formula (2) in this case reads

$$2p_i L_i = n_i h,$$

and therefore, the quantized energy levels have the form:

$$\mathcal{E}(\mathbf{n}) = \mathcal{E}\left(\frac{n_1 h}{2L_1}, \frac{n_2 h}{2L_2}, \frac{n_3 h}{2L_3}\right) \quad (8)$$

Performing the calculation according to the formulas of section 1 using (8) it is possible to get all terms of the oscillating part of the thermodynamic potential Ω (because of the two possible electron spin orientations, the right-hand side of the formula for Ω must be additionally multiplied by 2). It turns out that if we exclude the anomalously high anisotropy of the boundary surface $\mathcal{E}(\mathbf{p}) = \zeta$ then for $L_1 \approx L_2 \approx L_3 = L$ we can write

$$\frac{\mathcal{L}_{\mathbf{k}}}{P} \approx \frac{P}{Q} \approx \frac{L^{1/2}}{\left(\frac{\hbar \partial^2 \mathcal{E} / \partial p^2}{\partial \mathcal{E} / \partial p}\right)^{1/2}}$$

where the quantity $\frac{\hbar \partial^2 \mathcal{E} / \partial p^2}{\partial \mathcal{E} / \partial p}$ is equal to the de Broglie wavelength of an electron λ_B .

Considering $L \gg \lambda_B$ we can leave in the expression for (4) only the terms containing Ω_{osc}

Then

$$\Omega_{\text{osc}} = \Omega_1 + \Omega_2 + \Omega_3. \quad (9)$$

Here

$$\begin{aligned} \Omega_1 = & L_1 L_2 L_3 \frac{1}{\pi^2} \left\{ \sum_{\nu} \sum_{k_1=1}^{\infty} \beta^2(\mathbf{k}) \frac{\Psi(\lambda_{\mathbf{k}}) \cos(2\pi k_1 n_{\nu 1} \pm \pi/4 \pm \pi/4)}{\sqrt{|D(k_1, 0, 0)|}} + \right. \\ & \left. + \sum_{\nu'} \sum_{k_2=1}^{\infty} \dots + \sum_{\nu''} \sum_{k_3=1}^{\infty} \dots \right\}. \end{aligned}$$

The dots in the last two sums stand for expressions analogous to that one in the first sum; $\mathbf{n}_{\nu}(k_1)$ is the radius vector of a point of the boundary surface $E(\mathbf{n}) = \zeta$, at which the surface normal is parallel to the vector $(k_1, 0, 0)$. The vectors $\mathbf{n}(k_2)$ and $\mathbf{n}(k_3)$ and are defined in analogy.

$$\begin{aligned} \Omega_2 = & L_1 L_2 L_3 \frac{1}{\pi^2} \left\{ \sum_{\mu} \sum_{k_1 k_2=1}^{\infty} \beta^2(\mathbf{k}) \frac{\Psi(\lambda_{\mathbf{k}}) \cos(2\pi k_1 n_{\mu 1} + k_2 n_{\mu 2} \pm \pi/4 \pm \pi/4)}{\sqrt{|D(k_1, k_2, 0)|}} + \right. \\ & \left. + \sum_{\mu'} \sum_{k_1 k_3=1}^{\infty} \dots + \sum_{\mu''} \sum_{k_2 k_3=1}^{\infty} \dots \right\}, \end{aligned}$$

where $\mathbf{n}_{\mu}(k_1, k_2)$ is the radius vector of a point at which the normal to the boundary surface $E(\mathbf{n}) = \zeta$ is parallel to the vector $(k_1, k_2, 0)$. The vectors $\mathbf{n}_{\mu'}(k_1, k_3)$ and $\mathbf{n}_{\mu''}(k_2, k_3)$ are defined in analogy.

$$\Omega_3 = L_1 L_2 L_3 \frac{1}{(3\pi)^2} \sum_{\rho} \sum_{k_1 k_2 k_3=1}^{\infty} \beta^2(\mathbf{k}) \frac{\Psi(3\lambda_{\mathbf{k}}) \cos(2\pi \mathbf{k} \mathbf{n}_{\rho} \pm \pi/4 \pm \pi/4)}{\sqrt{|D(k_1, k_2, k_3)|}},$$

where $\mathbf{n}_{\rho}(k_1, k_2, k_3)$ is the radius vector of a point at which the normal to the boundary surface is parallel to the vector \mathbf{k} :

$$\lambda_{\mathbf{k}} = \frac{(2\pi)^2 \theta}{h\beta(\mathbf{k})};$$

$$D(k_1, k_2, k_3) = \begin{vmatrix} \partial^2 \mathcal{E} / \partial p_1^2 & \partial^2 \mathcal{E} / \partial p_1 \partial p_2 & \partial^2 \mathcal{E} / \partial p_1 \partial p_3 & K_1 L_1 \\ \partial^2 \mathcal{E} / \partial p_1 \partial p_2 & \partial^2 \mathcal{E} / \partial p_2^2 & \partial^2 \mathcal{E} / \partial p_2 \partial p_3 & k_2 L_2 \\ \partial^2 \mathcal{E} / \partial p_1 \partial p_3 & \partial^2 \mathcal{E} / \partial p_2 \partial p_3 & \partial^2 \mathcal{E} / \partial p_3^2 & k_3 L_3 \\ k_1 L_1 & k_2 L_2 & k_3 L_3 & 0 \end{vmatrix}.$$

The values of $\lambda_{\mathbf{k}}$ and $D(k_1, k_2, k_3)$ are taken at the corresponding points on the boundary surface.

The symbol $\beta(\mathbf{k})$ was used to denote the expressions

$$\frac{\partial \mathcal{E} / \partial p_1}{k_1 L_1} = \frac{\partial \mathcal{E} / \partial p_2}{k_2 L_2} = \frac{\partial \mathcal{E} / \partial p_3}{k_3 L_3} \beta(\mathbf{k}) \quad (10)$$

taken at the same points. Assuming $L_3 \gg L_1 \approx L_2$ the formula (9) for Ω_{osc} can be simplified. In the case of infinitely large L_3 , it follows from Eq. (10) that $k_3 = 0$ and hence Ω_3 in (9) vanishes and in the formulas for Ω_1 and Ω_2 only the terms with sums over k_1 and k_2 survive.

Finally, assuming $L_2 \approx L_3 \gg L_1$, i. e. in case of an electron gas confined between two infinitely high walls we get

$$\begin{aligned}\Omega_{\text{osc}} &= \Omega_1 = \\ &= \frac{L_2 L_3}{L_1^2} \frac{1}{\pi^2} \sum_{\nu} \sum_{k=1}^{\infty} \frac{(\partial \mathcal{E} / \partial p_1)^2 \Psi \left(\frac{4\pi^2 k L_1 \theta}{h \partial \mathcal{E} / \partial p_1} \right)}{\left| \frac{\partial^2 \mathcal{E}}{\partial p_2^2} \frac{\partial^2 \mathcal{E}}{\partial p_3^2} - \left(\frac{\partial^2 \mathcal{E}}{\partial p_2 \partial p_3} \right)^2 \right|^{\frac{1}{2}}} \cos \left(2\pi k n_{\nu_1} \pm \frac{\pi}{4} \pm \frac{\pi}{4} \right)\end{aligned}\quad (11)$$

The formulas given above for the oscillating part of Ω take the simplest form in case of an electron gas with a quadratic dispersion law

$$\mathcal{E} = \frac{p^2}{2m} \equiv \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

placed in a cubic potential well

$$(L_1 = L_2 = L_3 = L).$$

In this case the terms entering (9) have the form

$$\begin{aligned}\Omega_1 &= \frac{6\zeta}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^3} \Psi \left(2\sqrt{2}\pi^2 k \frac{\sqrt{m\theta L}}{h\sqrt{\zeta}} \right) \sin \left(4\pi k L \frac{\sqrt{2m\zeta}}{h} \right); \\ \Omega_2 &= \frac{6\zeta}{(2\pi)^2} \sum_{k_1, k_2=1}^{\infty} \frac{1}{(k_1^2 + k_2^2)^{3/2}} \Psi \left(4\sqrt{2}\pi^2 \sqrt{k_1^2 + k_2^2} \frac{\sqrt{m\theta L}}{h\sqrt{\zeta}} \right) \times \\ &\quad \times \sin \left(4\pi \sqrt{k_1^2 + k_2^2} L \frac{\sqrt{2m\zeta}}{h} \right); \\ \Omega_3 &= \frac{2\zeta}{(3\pi)^2} \sum_{k_1, k_2, k_3=1}^{\infty} \frac{1}{|\mathbf{k}|^3} \Psi \left(6\sqrt{2}\pi^2 |\mathbf{k}| \frac{\sqrt{m\theta L}}{h\sqrt{\zeta}} \right) \times \\ &\quad \times \sin \left(4\pi |\mathbf{k}| L \frac{\sqrt{2m\zeta}}{h} \right).\end{aligned}$$

If the temperature of the gas is very low or if the size of the potential well is very small, so that $\lambda_{\mathbf{k}} \equiv 2\sqrt{2}\pi^2 \frac{\sqrt{m\theta L}}{h\sqrt{\zeta}} \ll 1$, the function $\Psi(\lambda_{\mathbf{k}})$ can be replaced by unity. If the size of the potential well is large and $\lambda_{\mathbf{k}} \gg 1$, then $\Psi(\lambda_{\mathbf{k}}) \approx 2\lambda_{\mathbf{k}} e^{-\lambda_{\mathbf{k}}}$ and in formula (9) only the first term of the sum for Ω_1 survives.

$$\Omega_{\text{osc}} \approx 24\theta L \frac{\sqrt{2m\zeta}}{h} \exp \left\{ -2\sqrt{2}\pi^2 \frac{\sqrt{m\theta L}}{h\sqrt{\zeta}} \right\} \sin \left(4\pi L \frac{\sqrt{2m\zeta}}{h} \right).$$

It can be seen from the last expression that the variation of the linear size of the potential well leads to the oscillations of the thermodynamic potential Ω with the period

$$\Delta L = \frac{h}{2\sqrt{2m\zeta}}$$

which does not depend on L . Note that this period is identical with the half length of de Broglie wave for electrons at the boundary surface. As the number of the de Broglie halfwaves, $h/2\sqrt{2m\zeta}$, which can be placed between the potential walls determines the index of a boundary stationary state, each "vibration" of Ω is connected with a unit change of the number of levels lying below the boundary energy.

3 Electron gas in potential well in magnetic field

Consider the electron gas with an arbitrary dispersion law (7) in the same potential well as in section 2 placed in a homogenous magnetic field \mathbf{H} applied along the x_3 axis. If we choose the vector potential in the form $A_1 = -Hx_2$, $A_2 = A_3 = 0$, two of the conditions (2) are reduced to obvious relations:

$$2p_1L_1 = n_1h, \quad 2p_3L_3 = n_3h.$$

The third condition can be written in the form [4]:

$$S\left(E, \frac{n_1h}{2L_1}, \frac{n_3h}{2L_3}; H\right) = (n_2 + \gamma)\frac{ehH}{c}, \quad (12)$$

where $S(E, p_x, p_z; H)$ is the area of the intersection of the constant energy surface

$$\mathcal{E}(P_1, P_2, P_3) = E, \quad p_x - \frac{|e|H}{c}\frac{L_2}{2} < P_1 < p_x + \frac{|e|H}{c}\frac{L_2}{2}$$

by the plane $P_3 = p_z = \text{const.}$

The energy levels E are given by expression following from (12):

$$E = E(n_1, n_2, n_3; H).$$

If L_2 is small, one can suppose that the mean diameter of the classical electron orbit in the magnetic field is greater than L_2 . In such case, E is determined by different analytical expressions according to the relations between the diameter of the curve $\mathcal{E}(P_1, P_2, p_z) = E$, $p_z = \frac{n_3h}{2L_3} = \text{const.}$ and the quantity $\left|\frac{n_1h}{L_1} \pm \frac{eH}{c}L_2\right|$.

Employing (12) and performing the calculations according to the formulas of section 1 (taking into account the electron spin) we obtain all terms of the oscillating part of the thermodynamic potential Ω . A comparison of terms containing $\mathcal{L}_{\mathbf{k}}$, P and Q shows that for $L_1 \approx L_3 = L$ their ratio is equal to :

$$\frac{\mathcal{L}_{\mathbf{k}}}{P} \approx \frac{P}{Q} \approx \frac{\gamma L}{(ch/eH)^{1/2}},$$

where the quantity $(ch/eH)^{1/2}$ has a simple physical meaning: it is identical to the radius of the classical electron orbit in the magnetic field $r_0(H)$ corresponding to the lowest

energy level ($n_2 = 0$). Really, as for small p one can always use the quadratic expansion of the expression (7), the lowest energy levels of the electron motion in the magnetic field are of the form: $E_{n_2} = \left(n_2 + \frac{1}{2}\right) \frac{e\hbar H}{m^*c}$ where m^* is an effective mass, and therefore

$$r_0(H) = \frac{c\sqrt{2m^*E_0}}{eH} = \left(\frac{ch}{eH}\right)^{1/2}.$$

Assuming $L \gg r_0(H)$, we can leave as in 2 only those terms of Ω_{OSC} which contain $\mathcal{L}_{\mathbf{k}}$:

$$\Omega_{\text{OSC}} = \Omega^{(1)} + \Omega^{(2)} + \Omega^{(3)}$$

here

$$\Omega^{(1)} = L_1 L_3 \left(\frac{eH}{c}\right)^2 \frac{4}{\pi^2} \sum_m \sum_{k_2=1}^{\infty} \frac{\Psi(\lambda_{k_2}) \cos \left[\frac{2\pi k_2 c S_m(H)}{e\hbar H} \pm \frac{\pi}{4} \pm \frac{\pi}{4} - 2\pi k_2 \gamma \right]}{k_2^3 \Phi(\zeta, \mathbf{n}_m)},$$

where $\mathbf{n}_m(H)$ is the radius vector of a point at which $S(\zeta, p_x, p_z; H)$ has an extremal value $S_m(H)$ at constant ζ : $\left. \frac{\partial S}{\partial p_x} \right|_m = \left. \frac{\partial S}{\partial p_z} \right|_m = 0$; the normal to the surface

$$E(n_1, n_2, n_3; H) = \zeta \quad (13)$$

at this point is parallel to axis n_2 ;

$$\begin{aligned} \Omega^{(2)} = & L_1 L_3 \left(\frac{eH}{c}\right)^2 \frac{1}{\pi^2} \left\{ \sum_{\mu} \sum_{k_2=1}^{\infty} \frac{1}{k_2^3} \sum_{k_1=1}^{\infty} \frac{\Psi(2\lambda_{k_2}) \cos \left[2\pi(k_1 n_{\mu 1} + k_2 n_{\mu 2}) \pm \frac{\pi}{4} \pm \frac{\pi}{4} \right]}{\Phi[\zeta, \mathbf{n}_{\mu}(k_1, k_2)]} + \right. \\ & \left. + \sum_{\mu'} \sum_{k_2=1}^{\infty} \frac{1}{k_2^3} \sum_{k_3=1}^{\infty} \frac{\Psi(2\lambda_{k_2}) \cos \left[2\pi(k_3 n_{\mu' 3} + k_2 n_{\mu' 2}) \pm \frac{\pi}{4} \pm \frac{\pi}{4} \right]}{\Phi[\zeta, \mathbf{n}_{\mu'}(k_3, k_2)]} \right\}; \end{aligned}$$

where $\mathbf{n}_{\mu}(k_1, k_2, H)$ is the radius vector of a point at which the normal to the surface (13) is parallel to the vector $(k_1, k_2, 0)$ and $\mathbf{n}_{\mu'}(k_3, k_2, H)$ is defined analogously;

$$\Omega^{(3)} = L_1 L_3 \left(\frac{eH}{c}\right)^2 \left(\frac{2}{3\pi}\right)^2 \sum_{\rho} \sum_{k_2=1}^{\infty} \frac{1}{k_2^3} \sum_{k_1, k_3=1}^{\infty} \frac{\Psi(3\lambda_{k_2}) \cos \left[2\pi \mathbf{k} \mathbf{n}_{\rho}(\mathbf{k}) \pm \frac{\pi}{4} \pm \frac{\pi}{4} \right]}{\Phi[\zeta, \mathbf{n}_{\rho}(\mathbf{k})]}$$

where $\mathbf{n}_{\rho}(\mathbf{k}, H)$ is the radius vector of a point at which the normal to the surface (13) is parallel to the vector \mathbf{k} ;

$$\lambda_{k_2} = 2\pi^2 k_2 \frac{c\theta}{e\hbar H} \left(\frac{\partial S}{\partial \zeta}\right), \quad \Phi(\zeta, \mathbf{n}) = \left| \frac{\partial S}{\partial \zeta} \left| \frac{\partial^2 S}{\partial p_x^2} \frac{\partial^2 S}{\partial p_z^2} - \left(\frac{\partial^2 S}{\partial p_x \partial p_z}\right)^2 \right|^{1/2} \right|$$

The values λ_{k_2} and $\Phi(\zeta, \mathbf{n})$ are taken at the corresponding points.

It should be noted that if some of the enumerated points lies in the coordination plane of the \mathbf{n} -space, the corresponding term must be multiplied by $\frac{1}{2}$; if this point lies in one of the coordination axes, it is necessary to multiply the corresponding term by $\frac{1}{4}$.

The formula for Ω_{osc} simplifies in two limiting cases: If $\lambda_{k_2} \ll 1$ then the function $\Psi(\lambda_{k_2})$ can be replaced by unity; if the $\lambda_{k_2} \gg 1$, then $\Psi(\lambda_{k_2}) \approx 2\lambda_{k_2}e^{-\lambda_{k_2}}$ and therefore only the first terms remain in the sums over \mathbf{k} . The last case corresponds to a strengthened inequality

$$\frac{c\theta}{ehH} \left(\frac{\partial S}{\partial \zeta} \right) \gg 1 \quad (14)$$

which has a simple physical meaning. As follows from (12), the mean distance between neighbouring energy levels with quantum numbers n_2 and n_2+1 near the boundary energy is given by

$$\Delta E_{n_2} = \frac{ehH}{\frac{c}{\frac{\partial S}{\partial \zeta}}}$$

and therefore the inequality (14) corresponds to

$$\zeta \gg \theta \gg \Delta E_{n_2}$$

If this inequality is fulfilled, Ω_{osc} consists of terms of the form

$$\Omega_{\text{osc}} \sim B(L_1, L_2, L_3; H, \zeta) \cos [2\pi n(L_1, L_2, L_3; H, \zeta) \pm \pi/4 \pm \pi/4]$$

The amplitude of the oscillations B and their phases $2\pi n$ have for $\Omega^{(j)}$ ($j = 1, 2, 3$) the following forms:

$$\begin{aligned} B(L_1, L_2, L_3; H, \zeta) &= L_1 L_2 \frac{16}{j} \frac{eH\theta}{ch} \left(\frac{\partial S}{\partial \zeta} \right) \frac{\exp\left\{-j2\pi^2 \frac{c\theta}{ehH} \frac{\partial S}{\partial \zeta}\right\}}{\Phi(\zeta, \mathbf{n})} & (j = 1, 2, 3) \\ n &= \frac{cS_m(H)}{ehH} - 2\pi\gamma & \text{for } j = 1, \\ n &= n_{\mu 1}(L_1, L_1, L_3; H, \zeta) + n_{\mu 2}(L_1, L_2, L_3; H, \zeta) & \text{for } j = 2, \\ n &= n_{\rho 1}(L_1, L_1, L_3; H, \zeta) + \\ &+ n_{\rho 2}(L_1, L_1, L_3; H, \zeta) + n_{\rho 3}(L_1, L_1, L_3; H, \zeta) & \text{for } j = 3. \end{aligned}$$

The dependence of the phase $2\pi n$ on the intensity of the magnetic field H and on the sizes of the potential well L_1, L_2, L_3 determine the periods of oscillations of Ω_{osc} with changes of these parameters.

It is easy to prove that the oscillation periods of the terms of Ω_{osc} with the changes of H are given by the following formulas:

$$\begin{aligned} 1) \quad \text{for } \Omega^{(1)} & \quad \Delta \left(\frac{1}{H} \right) = \frac{eh/c}{|S_m(H) - H\partial S_m/\partial H|}, \\ 2) \quad \text{for } \Omega^{(2)} & \quad \Delta \left(\frac{1}{H} \right) = \frac{eh/c}{2|S(\mathbf{n}_\mu, H) - H\partial S_\mu/\partial H|}, \\ 3) \quad \text{for } \Omega^{(3)} & \quad \Delta \left(\frac{1}{H} \right) = \frac{eh/c}{3|S(\mathbf{n}_\rho, H) - H\partial S_\rho/\partial H|}. \end{aligned}$$

In a similar way, it is possible to obtain the periods of oscillations with the changes of L_1, L_2, L_3 .

Supposing that $L_3 \gg L_1$ then $k_3 = 0$ and the following formula is valid for Ω_{osc} :

$$\begin{aligned} \Omega_{\text{osc}} &= \Omega_1 + L_1 L_2 \left(\frac{eH}{c} \right)^2 \frac{1}{\pi^2} \sum_{\mu} \sum_{k_2=1}^{\infty} \frac{1}{k_2^3} \sum_{k_1=1}^{\infty} \Psi(2\lambda_{k_2}) \times \\ &\times \frac{\cos \left[2\pi(k_1 n_{\mu 1} + k_2 n_{\mu 2}) \pm \frac{\pi}{4} \pm \frac{\pi}{4} \right]}{\Phi[\zeta, \mathbf{n}_\mu(k_2, k_3)]}. \end{aligned}$$

Finally, assuming that $L_1, L_3 \gg L_2$, the formula for Ω_{osc} consists only of the first term (Ω_1):

$$\begin{aligned} \Omega_{\text{osc}} &= \Omega_1 = L_1 L_3 \left(\frac{eH}{c} \right)^2 \frac{4}{\pi^2} \times \\ &\times \sum_m \sum_{k_2=1}^{\infty} \frac{\Psi(\lambda_{k_2}) \cos \left[\frac{2\pi k_2 c S_m(H)}{ehH} \pm \frac{\pi}{4} \pm \frac{\pi}{4} - 2\pi k_2 \gamma \right]}{k_2^3 \Phi(\zeta, \mathbf{n}_m)} \end{aligned} \quad (15)$$

which coincides with the formula derived by the present authors for the case of a one-dimensional infinitely deep potential well [4] in a slightly different way¹.

It is not difficult to verify that in the limiting case $H \rightarrow 0$ the formula (15) transforms in (11).

References

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¹In [4] a formula for the oscillating part of the magnetic moment of an electron gas M_{osc} was obtained. Using $M = -\partial\Omega/\partial H$ and employing the fact that the prefactor of cosine in (15) need not be differentiated, it is easy to prove the equivalence of both formulas.